Stability of rarefaction for stochastic viscous conservation law

Zhao Dong

Joint work with Feimin Huang and Houqi Su

Academy of Mathematics and Systems Science, CAS

11.25-27, 2022

Catalogue

1. Introduction
   - Background
   - Known results
   - Main results
2. H^1 global well-posedness of perturbation equation (11)
   - Monotone type property of the lnearized cut-off equation (19)
   - Convergence of the cut-off solutions
3. H^2 regularity of the perturbation equation (11)
   - Small inital value gives H^2 regularity
4. Decay estimates
   - L^p decay for general H^1 initial value, $2 \leq p < \infty$
   - L^∞ decay for small H^1 initial value

The one dimensional viscous conservation law

$$du + f(u)_x dt = \nu u_{xx} dt,\ \nu > 0. \tag{1}$$

when $\nu = 0$, the equation becomes conservation law

$$du + f(u)_x dt = 0, \tag{2}$$

which admits rich wave phenomena such as shock and rarefaction wave.

For systems of viscous conservation laws:

$$U_t + F(U)_x = \nu(B(U)U_x)_x, U \in R^n, \tag{3}$$

which includes 1-d compressible Navier-Stokes equations,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \nu u_{xx}. \end{cases} \tag{4}$$

When $\nu = 0$, the system (3) becomes

$$U_t + F(U)_x = 0 \tag{5}$$

and the NS system is reduced to the compressible Euler equation

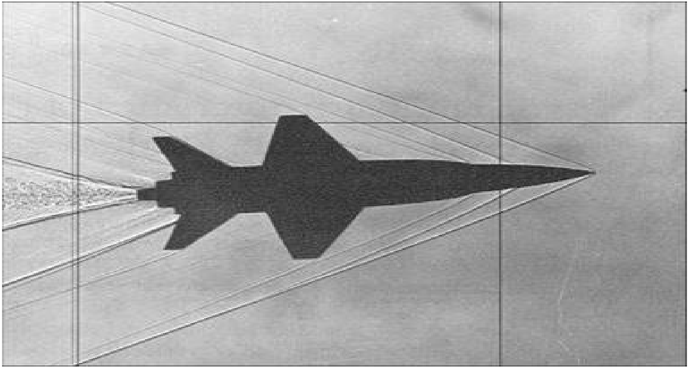
$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \end{cases} \tag{6}$$

which has important applications in the field of gas dynamics.

Consider the system (6) with Riemann initial data

$$(\rho, \rho u)(x, 0) = \begin{cases} (\rho_-, \ \rho_- u_-) & x < 0, \\ (\rho_+, \ \rho_+ u_+) & x > 0. \end{cases} \tag{7}$$

L. Riemann first considered the Euler equation (6) with such kind of initial data in 1860 and gave explicit formula of shock and rarefaction wave.



shock wave (download from google)

Let's go back to the conservation law (2) with the Riemann initial data

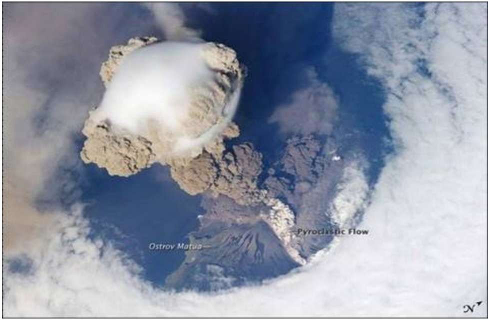
$$u(x,0) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{8}$$

If $u_- < u_+$, the solution of (2) is rarefaction wave,

$$u^r(t,x) = \begin{cases} u_-\,, & \frac{x}{t} < f'(u_-)\,, \\ (f')^{-1}\left(\frac{x}{t}\right), & f'(u_-) < \frac{x}{t} < f'(u_+)\,, \\ u_+\,, & \frac{x}{t} > f'(u_+)\,. \end{cases}$$

which does not vanish as $t \to +\infty$.

Rarefaction wave in real life, see the volcanic eruption as the image below



(download from google)

Riemann solution

- Riemann 1860, P.D.Lax, 1957(general hyperbolic conservation laws), Riemann solution is the linear superposition of shock, rarefaction and contact discontinuity (linear wave)
- Riemann solutions govern both local and long time behavior
- Building block to approximate the solutions of Cauchy problem for strictly hyperbolic systems of conservation law, Riemann solver

Well posedness of small BV solution through Riemann solution:

- J. Glimm, (CPAM, 1965, Glimm Scheme, global existence)
- Bressan et.al, Liu-Yang, P. Le Floch, $\cdots$ (uniqueness and stability)
- Bressan-Bianchini (Ann.of Math. 2004, vanishing viscosity)
- Bressan-Yang (CPAM 2004, convergence rate), 2×2 Euler equations
- Bressan-Huang-Wang-Yang (SIMA 2012, convergence rate), 3×3 Euler equation

Problem: It is important to study the properties of Riemann solutions such as its time-asymptotic stability.

Due to the effect of viscosity, it is commonly conjectured that the large time behavior of the solutions to the viscous conservation laws (3) is governed by a viscous version of the Riemann solution of the corresponding inviscid system of conservation laws.

Rrarefaction wave are stable for Burgers equation

Ilin and Oleinik (1964): first showed

The stability of rarefaction waves :

$$\lim_{t\to\infty} \left\| u(t,x) - u^r(t,x) \right\|_{\mathbb{L}^\infty(\mathbb{R})} = 0.$$

Hattori and Nishihara (1991): decay rate

$$\left\| u(t,x) - u^r(t,x) \right\|_{\mathbb{L}^p(\mathbb{R})} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \forall p \in (1,\infty].$$

In their work, $u - u^r \in \mathbb{L}^1(\mathbb{R})$ is essentially assumed.

Rarefaction wave is stable for NS equation

Rarefaction wave

- Matsumura-Nishihara, Japan. Jour. App. Math., 1986
- Liu-Xin, CPAM, 1988
- Matsumura-Nishihara, CMP,1992
- Nishihara-Yang-Zhao, SIMA,2004
- $\cdots$

Stochastic viscous conservation law

Question: Would rarefaction waves be still stable under the stochastic perturbation?

We consider the following stochastic conservation law (SVCL) with infinite dimensional conservative noise.

$$du + \partial_x f(u)dt = \partial_x \left[b(u)u_x + \sum_{i=1}^{\infty} \sigma_i(u) \circ dB_i(t) \right]. \tag{9}$$

where $f''(u) \geq \alpha > 0$ and $b(u) \geq \mu > 0$ for all $u \in \mathbb{R}$. $\{B_i(t)\}_{i=1}^{\infty}$ are i.i.d one-dimensional standard Brown motions on probability space $(\Omega, \mathcal{F}, P)$.

The conservative noise keeps the conservation structure of the equation, and has drawn attention recently years.

To demonstrate the idea of the conservative noise, we give a simple example of stochastic conservation law

$$u_t + f(u)_x = \sigma(u)_x \circ \dot{B}(t), x \in \mathbb{R} \tag{10}$$

Assume $u(t, a) = u(t, b)$, integrating (10) over a given interval (a, b) one obtains

$$\begin{aligned}
&\frac{d}{dt}\int_a^b u(t,x)dx = \int_a^b -f(u(t,x))_x + \sigma(u)_x \circ \dot{B}(t)dx \\
&= [f(u(t,a)) - \sigma(u(t,a)) \circ \dot{B}(t)] - [f(u(t,b)) - \sigma(u(t,b)) \circ \dot{B}(t)] \\
&= [\text{ inflow at } a] - [\text{ outflow at } b] = 0.
\end{aligned}$$

That is, u is neither created nor destroyed: the total amount of u contained inside (a, b) is conseved due to the stochastic flow $f - \sigma \circ \dot{B}(t)$ of u across boundary points coincides.

Well-possedness results for the SCL/SVCL

SVL $\mu = 0$

- Kim, Indian. Univ. Math. Jour., 2003, Entropy solution, unbounded domain in 1d
- J. Feng, et al., Jour. Funct. Analy., 2008, Entropy solution, unbounded domain in general dimensions
- A. Debussche , J. Vovelle, Jour. Funct. Analy., 2010, Kinetic solutions,in $\mathbb{T}^1$
- $\cdots$

SVCL $\mu > 0$

- Stocia, et al., Elec. Jour. Prob., 2004, Semilinear and homogeneous diffusion operator
- A. Debussche, et al., Ann. Prob., 2013, Kinetic solution, in $\mathbb{T}^1$
- Zhang et al., Stoch. Proc. Appl., 2017, Strong solution, in $\mathbb{T}^1$
- Z. Dong, F. Huang, H. Su, 2022, Burgers equation in $\mathbb{R}^1$, Strong solution
- $\cdots$

Long time behaviors for the SCL/SVCL

SCL $\mu = 0$

- E. W, et al., Ann. Math., 2000, Burgers equation in $\mathbb{T}^1$, invariant measure and main shock
- Y. Bakhtin, Ann. Prob., 2013, Burgers equation in $\mathbb{R}^1$, Basin of attraction
- ...

SVCL $\mu > 0$

- B. Gess, P. Souganidis, CPAM, 2017, Invariant measures and regularizing effects in $\mathbb{T}^1$
- Y. Bakhtin, et al., CPAM, 2019, Thermodynamic Limit, Stationary solutions in $\mathbb{R}^1$
- B. Gess, P. Souganidis, Stoch. Proc. Appl., 2020, Long time averaging in $\mathbb{T}^1$
- Z. Dong, F. Huang, H. Su, 2021, Burgers equation in $\mathbb{R}^1$, Stabal for rarefaction wave/instable for shock wave
- A. Dunlap, L, Ryzhik, CMP, 2021, ARMA, 2021, Invariant measure of the Burger equation in $\mathbb{R}^1$, stability of the shock wave
- ...

Other models related to the SVLs

- M. Gubinelli, M. Jar, Stoch. Partial Differ. Equ. Anal. Comput. 2013, Regularization by noise
- M. Hairer, H. Weber, Probab. Theory & Rel. Fields, 2013, Rough Burgers-like equations
- P. Goncalves, M. Jara, S. Sethuraman, Ann. Probab., 2015, Microscopic derivation of a fractional stochastic Burgers equation
- B. Fehrman, B. Gess, Arch. Rational Mech. Anal., 2019, Well-Posedness of Nonlinear Diffusion Equations with Nonlinear, Conservative Noise
- L. Galeati, Stoch PDE: Anal Comp, 2020, Convergence of transport noise to a deterministic parabolic equation
- A. Dunlap, L. Ryzhik, Viscous shock solutions to the stochastic Burgers equation, ARMA, 2021,
- $\cdots$

Our set up

We focus on the well-possedness and the stability of the SVCL (9) towards the rarefaction wave.

Let $\phi = u - \bar{u}$, $\bar{u}$ is the approximate rarefaction wave. The perturbed equation is

$$\begin{cases} d\phi + [f(\phi + \bar{u}) - f(\phi)]_x dt \\ = \partial_x \left[\left(b(\phi + \bar{u}) + \frac{1}{2} \sum_{i=1}^{\infty} \sigma_i'^2 (\phi + \bar{u}) \right) (\phi + \bar{u})_x \right] dt + \partial_x \left[\sum_{i=1}^{\infty} \sigma_i (\phi + \bar{u}) \cdot dB_i(t) \right], \\ \phi(0) = u(0) - \bar{u}(0) = \phi_0. \end{cases} \tag{11}$$

This is a qausi-linear SPDE with infinite dimensional conservative noise.

Answer: the rarefaction wave is still stable under conservative noise.

In the following, we set $L^p_r := L^p(B(r))$ and $H^s_r := H^s_0(B(r))$. We shall omit the natural embedding from H^s_r into $H^s(\mathbb{R})$ in the later context. Let X be a metric sapce with metric $\|\cdot\|_X$, we set

$$\|g\|_{l^p(X)} := \left(\sum_{i=1}^{\infty} \|g_i\|_X^p\right)^{\frac{1}{p}}, \quad g \in l^p(X).$$

We assume that σ' is locally $l^2(H^1)$ bounded

(H) For any $N > 0$, $\sum_{i=1}^{\infty} \|\sigma_i'\|^2_{H^1(-N,N)} < \infty$.

Main results

Theorem 1 (Global well-possedness and L^p decay)

Assume (H). For all $\phi_0 \in H^1(\mathbb{R})$, we have a unique strong solution $\phi \in X_1(T)$ solves the equation (11) *with the following decay estimate, for $2 \leq p < \infty$*

$$\mathbb{E}\|\phi\|^p_{\mathbb{L}^p(\mathbb{R})} \leq C_p(2+t)^{-\frac{p-2}{4}} \ln^{\frac{p}{2}}(2+t). \tag{12}$$

For small initial value, we have derivative and L^∞ decay estimate

Theorem 2 (H^2 regularity, derivative and L^∞ decay)

Assume (H). For all $\phi_0 \in H^2(\mathbb{R})$, set $M := \|\phi_0\|_{L^\infty(\mathbb{R})} + u_+ - u_-$, if

$$\|\phi_0\|_{L^\infty(\mathbb{R})}\left(\|b'\|_{L^\infty_M} + 3\|\sigma''\|^2_{l^2(L^2_M)}\right) \leq \frac{1}{12} \inf_{|x|\leq M} b(x), \tag{13}$$

then we have a unique $\phi \in X_2(T)$ solves the equation (11) *with the following decay estimates*

$$\begin{aligned} \mathbb{E}\|\phi_x\|^2 &\leq C(1+t)^{-1} \ln(2+t), \\ \mathbb{E}\|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} &\leq C(2+t)^{-\frac{1}{4}} \ln^{\frac{1}{2}}(2+t). \end{aligned} \tag{14}$$

Remark 1

For linear σ and constant b as in our previews work, condition (40) is natrually satisfied.

Remark 2

For the heat equation $u_t = u_{xx},\ u(0,x) \in L^2(\mathbb{R})$, the optimal decay rate of $u(t,x)$ in $\mathbb{L}^p$ is $(1+t)^{-\frac{p-2}{4p}}$. In this sense, the decay rate (14) is almost optimal! In fact, the term $\ln(2+t)$ in (14) is coming from the Brownian motions $B_i(t)$.

Remark 3

In 2021, A. Dunlap and L. Ryzhik studied Viscous shock solutions to the stochastic Burgers equation (SBE), where noise is taken as the gradient of smoothed white noise.

They constructed the stochastic shock profile as some function connects two ordered space-time stationary solutions of the SBE.

They proved that the solution of the SBE is governed by the stochastic shock asymptotically.

Main methods for Theorem 1 and Theorem 2

- For the well-possedness, the mild approach can't be implemented since, the parabolic term is inhomogeneous and quasilinear. Nevertheless, the operaters are not monotone and the domain is unbounded, which leads to no uniform bounds. We need to linearize the equation deduce the maximal principle and modify the monotone trick to apply in the converge analysis H^1 locally.

- Classial L^2 argument fails: $\mathbb{E}||\phi(t,\cdot)||_{\mathbb{L}^2(\mathbb{R})}$ may increase with time t, while it is uniformly bounded for the deterministic case. However, for any $p \in (2, +\infty)$, $\mathbb{E}||\phi(t,\cdot)||_{\mathbb{L}^p(\mathbb{R})}$ decays by a new L^p energy method and BDG inequality.

- The L^∞ decay is based on the dereivative estimates, which requires higher regularity of the strong solution. The difficult term ϕ_x^4 makes the smallness of the initial value inevitable.

H^1 global well-posedness of perturbation equation (11)

For $i = 1, 2$, define the solution spaces

$$X_i(T) := L^2\left([0,T]\times\Omega, dt\otimes d\mathbb{P}, H^i(\mathbb{R})\right)\bigcap C\left([0,T]\times\Omega, dt\otimes d\mathbb{P}, H^{i-1}(\mathbb{R})\right),$$

We first solve a sequence of linearized cut off equation of initial value problems in $B(N) = (-N, N)$. Using the modified "locally" monotone trick and maximal principle, we can argue that the weak limit satisfies the equation (11).

Cut off of equation (11)

For $B(N)$, we define the following truncations

1. $$f_N \in C^2,\ \ \text{supp}\, f_N \subseteq [-2N, 2N],\ \ \text{supp}(f_N - f) \subseteq [-N, N]^c,$$
$$\lim_{N\to\infty} f_N' = f' \text{ in } L^\infty_{loc}(\mathbb{R}), \quad \lim_{N\to\infty} f_N'' = f'' \text{ in } L^1_{loc}(\mathbb{R}), \tag{15}$$

2. $$b_N \in C^1,\ \ \text{supp}\, b_N \subseteq [-2N, 2N],\ b_N = b \text{ in } [-N, N],\ b_N = \mu \text{ in } [-2N, 2N]^c,$$
$$\lim_{N\to\infty} b_N = b \text{ in } L^\infty_{loc}(\mathbb{R}), \tag{16}$$

3. $$\sigma_{i,N} \in C^2,\ \ \text{supp}\sigma_{i,N} \subseteq [-2N, 2N],\ \ \text{supp}(\sigma_{i,N} - \sigma_i) \subseteq [-N, N]^c,$$
$$\lim_{N\to\infty} \sigma_N' = \sigma' \text{ in } l^2(H^1_{loc}), \tag{17}$$

4. $$\phi_0^{(N)} \in H^1_N,\ \|\phi_0^{(N)}\|_{L^\infty_N} \le \|\phi_0\|_{L^\infty(\mathbb{R})},$$
$$\lim_{N\to\infty} \phi_0^{(N)} = \phi_0 \text{ in } H^1(\mathbb{R}). \tag{18}$$

Then we have the cut-off equation in H_N^{-1}

$$\begin{cases} d\phi^{(N)} = A_N(\phi^{(N)})dt + B_N(\phi^{(N)})dt + \sigma_N(\phi^{(N)})dQ_N(t) \\ \phi^{(N)}(0) = \phi_0^{(N)}, \end{cases} \tag{19}$$

where

$$\int_0^T \sigma_N(v)dQ_N(t) = \int_0^T \sum_{i=1}^N \sigma'_{i,N}\left(\phi^{(N-1)} + \bar{u}\right)(v_x + \bar{u}_x)dB_i(t)$$

$$:= \int_0^T \sum_{i=1}^N \sigma'_{i,N}(v_x + \bar{u}_x)dB_i(t),$$

where we use $\sigma'_{i,N} := \sigma'_{i,N}(\phi^{(N-1)} + \bar{u})$ for short,

$$A_N(v) = \partial_x\left[\left(b_N\left(\phi^{(N-1)} + \bar{u}\right) + \frac{1}{2}\sum_{i=1}^N \sigma'^2_{(N),i}\left(\phi^{(N-1)} + \bar{u}\right)\right)(v_x + \bar{u}_x)\right] \tag{20}$$

$$:= \partial_x\left[\left(b_N + \frac{1}{2}\sum_{i=1}^N \sigma'^2_{i,N}\right)(v_x + \bar{u}_x)\right],$$

where we use $b_N := b_N(\phi^{(N-1)} + \bar{u})$ for short,

$$B_N(v) = -\partial_x\left[f_N(v + \bar{u}) - f_N(\bar{u})\right].$$

Define the cut off solution space

$$X_i^{(N)}(T) := L^2\left([0,T]\times\Omega, dt\otimes d\mathbb{P}, H_N^i\right)\bigcap C\left([0,T]\times\Omega, dt\otimes d\mathbb{P}, H_N^{i-1}\right), \quad i = 1, 2$$

Proposition 3 (Well-posedness of the cut-off)

Assume (H), then the equation (19) *satisfies the conditions in the weakly monotone theory.*

- *(Hemicontinuity) For all* $u, v, w \in V, \omega \in \Omega$ *and* $\lambda \in [0, T]$ *the map*

$$\mathbb{R} \ni \lambda \mapsto_{H_N^{-1}} \langle (A_N + B_N)(t, u + \lambda v), w \rangle_{H_N^1} \tag{21}$$

  is continuous.

- *(Weak monotonicity) There exists* $c \in \mathbb{R}$ *such that for all* $u, v \in V$

$$\begin{aligned} &2\left\langle (A_N + B_N)(u) - (A_N + B_N)(v), u - v \right\rangle + \langle \sigma_N(t, u)Q_N(t) - \sigma_N(t, v)Q_N(t) \rangle \\ \leq & -\mu \|u_x - v_x\|^2 + 2C_{N,\mu}\|u - v\|^2. \end{aligned} \tag{22}$$

- *(Coercivity)*

$$\begin{aligned} &2\left\langle (A_N + B_N)(u), u \right\rangle + \langle \sigma_N(t, u)Q_N(t) \rangle \\ \leq & -\mu \|u_x\|^2 + C_{\mu,\sigma}\|\bar{u}_x\|^2 \leq -\mu\|u_x\|^2 + C_{\mu,\sigma}(1 + t)^{-1}. \end{aligned} \tag{23}$$

- *(Boundedness)*

$$\|A_N(u) + B_N(u)\|_{H_N^{-1}} \leq C_{N,\sigma}\|u\|_{H_N^1} \text{ on } \Omega. \tag{24}$$

By the weakly monotone method (Liu-Roeckner), the equation (19) *has a unique solution* $\phi^{(N)}$ *in* $X_1^{(N)}(T)$ *for all* $T > 0$.

In order to get the well-possedness and decay results, we need the following a priori estimates including an useful locally maximal principle.

Lemma 4

Assume (H) and $\phi^{(N)}(t,x) \in X_T^{(N)}$ be the strong solution of (19), then we have L^2 estimate

$$\|\phi^{(N)}(t)\|^2 + \int_0^t \|\phi_x^{(N)}\|^2 ds + \int_0^t \int_{B(N)} \phi^{(N)2} \bar{u}_x dx\, ds \\ \leq C_1 \left[(1 + \|\phi_0\|^2) + \ln(1+t) \right] + C_2 \int_0^t \int_{B(N)} \sum_{i=1}^N \sigma_i (\phi^{(N)} + \bar{u})_x \phi^{(N)} dx\, dB_i(t). \tag{25}$$

L^p estimates for $p > 2$

$$\|\phi^{(N)}(t)\|_{L_N^p}^p + \int_0^t \int_{B(N)} |\phi^{(N)}|^p \bar{u}_x dx\, ds + \int_0^t \int_{B(N)} |\phi^{(N)}|^{p-2} \phi_x^{(N)2} dx\, ds \\ \leq pC \int_0^t (1+s)^{-\frac{p}{2}} ds + pC \int_0^t (1+s)^{-1} \|\phi^{(N)}\|_{L_N^p}^p ds + pdM(t). \tag{26}$$

For $p = \infty$

$$\sup_{0 \leq t \leq T} \|\phi^{(N)}(t)\|_{L_N^\infty} \leqslant \|\phi_0\|_{L^\infty(\mathbb{R})}, \quad a.s. \tag{27}$$

Convergence to the perturbation equation (11)

To prove the weak limite of $\phi^{(N)}$ is the strong solution ϕ of equation (11), we need the following crucial lemma of the "local" property

Lemma 5 (Localize the operators uniformly in N)

For $R > 0$, there exists $C_R > 0$ such that for all N and all $u, v \in B_{\sqrt{2}R}(H^1(\mathbb{R}))$, almost surely we have

$$\|A_N(u) + B_N(u)\|_{H_N^{-1}} \leq C_R \|u\|_{H_N^1}, \tag{28}$$

$$\begin{aligned} &\langle (A_N + B_N)(u - v), u - v\rangle + \langle (\sigma_N(t, u) - \sigma_N(t, v))\, Q_N(t)\rangle \\ &\leq -\mu\|u_x - v_x\|^2 + 2C_R\|u - v\|^2. \end{aligned} \tag{29}$$

Global well-possedness of perturbation equation (11)

Lemma 6 (Global well-possedness of equation (11))

Assume (H), for any $T > 0$, we have unique $\phi \in X_1(T)$ solves the equation (11) *with the following estimate*

$$\mathbb{E}\sup_{0\le s\le T}\left\|\phi(s)\right\|^2 + \mu\mathbb{E}\int_0^T \|\phi_x\|^2\,ds \le \|\phi_0\|^2 + C_\sigma \ln(1+T). \tag{30}$$

Sketch of the proof.

Existence: Utilizing lemma 5 we can manipulate the classic weak monotone methods inside an uniform ball in $H^1(\mathbb{R})$, where the radius is dertermined by the initial value thanks to the maximal principle in lemma 4.
Uniqueness: take ϕ_1, ϕ_2 be two strong solutions of the perturbation equation (11) in $X_1(T)$. Since the equation (11) does not satisfy the monotone condition, we can use the L^p frame to obtain the maximal principle for $\psi = \phi_1 - \phi_2$, which leads to the uniqueness. □

By the same argument we can also obtain the priori estimates of the solution ϕ of (11).

Lemma 7

Assume the conditions (H) and $\phi(t,x) \in X_1(T)$ *be the strong solution of* (11), *then for any* $T > 0$ *we have we have* L^2 *estimate*

$$\begin{aligned} &\|\phi(t)\|^2 + \int_0^t \|\phi_x\|^2 ds + \int_0^t \int_{\mathbb{R}^n} \phi^2 \bar{u}_x dxds \\ &\leq C_1 \left[(1 + \|\phi_0\|^2) + \ln(1+t)\right] + C_2 \int_0^t \int_{\mathbb{R}} \sum_{i=1}^{N} \sigma_i(\phi + \bar{u})_x \phi dx\, dB_i(t). \end{aligned} \tag{31}$$

For $p > 2$

$$\begin{aligned} &\|\phi(t)\|^p_{L^p(\mathbb{R})} + \int_0^t \int |\phi|^p \bar{u}_x dxds + \int_0^t \int_{\mathbb{R}} |\phi|^{p-2} \phi_x^{(N)2} dxds \\ &\leq pC \int_0^t (1+s)^{-\frac{p}{2}} ds + pC \int_0^t (1+s)^{-1} \|\phi\|^p_{L^p(\mathbb{R})} ds + pdM(t). \end{aligned} \tag{32}$$

For $p = \infty$

$$\sup_{0 \leq t \leq T} \|\phi(t)\|_{L^\infty(\mathbb{R})} \leqslant \|\phi_0\|_{L^\infty(\mathbb{R})}, \qquad a.s.. \tag{33}$$

H^2 regularity of the equation (11)

In order to obtain the L^∞ decay, we need higher regularity of the strong solution ϕ. For this reason, we implement the derivative estimate in the Galerkin approximation.

Take $\{e_k\}_{k=1}^\infty$, which are consists of $\sin(kx)$ and $\cos(kx)$, be orthognal basis of L^2_N and $\text{span}\{e_1, \cdots, e_k, \cdots\} \overset{dense}{\subseteq} H^2_N$. Define P_k

$$P_k^N : H_N^{-2} \to H_N^2, \quad P_k^N u := \sum_{j=1}^{k} \langle u, e_j \rangle e_j,$$

let $H_k^N := \text{Im}\left(P_k^N|_{L^2_N}\right)$.

Notice that P_k^N and ∂_x commutes in H^1_N.

H^2 regularity of the equation (11)

Consider the following SDE in Garlerkin approximation of the cut off equation (19)

$$\begin{cases} d\phi_k^{(N)} = A_{k,N}\left(\phi_k^{(N)}\right)dt + B_{k,N}\left(\phi_k^{(N)}\right)dt + \sigma_{k,N}\left(\phi_k^{(N)}\right)dQ_N(t) \\ \phi_k^{(N)}(0) = P_k\phi_0. \end{cases} \tag{34}$$

For $v \in H_k$.

$$\begin{aligned} A_{k,N}(v) &:= P_k^N A_N v = P_k^N \partial_x \left[\left(b_N + \frac{1}{2}\sum_{i=1}^{N} \sigma_{i,N}'^2\right)(v_x + \bar{u}_x)\right], \\ B_{k,N}(v) &:= P_k^N B_N(v) = -P_k^N \partial_x \left[f_N(v+\bar{u}) - f_N(\bar{u})\right], \\ \sigma_{k,N}(v)\,dQ_N &:= P_k^N \sigma_N(v)\,dQ_N = P_k^N \sum_{i=1}^{N} \sigma_{i,N}'(v_x + \bar{u}_x)dB_i(t). \end{aligned} \tag{35}$$

H^2 regularity of the equation (11)

Define the stopping time

$$\tau_k^{(N)}(R) := \inf\left\{t \geq 0, \left\|\phi_k^{(N)}(t)\right\|_{L^\infty(\mathbb{R})} \geq R\right\}.$$

Lemma 8 (A priori derivative estimate before a stopping time)

Assume (H). Take $R > \delta := u_+ - u_-$, for $t \in [0, \tau_k^{(N)}(R-\delta) \wedge \tau_{k-1}^{(N)}(R-\delta)]$, we have the growth estimates of $\phi^{(N)}(t)$

$$\mathbb{E}\left[\left\|\phi_{k,x}^{(N)}(T)\right\|^2 + \int_0^T \int_{\mathbb{R}} \left[b_N - 12R\left(\|b_N'\|_{L^\infty(\mathbb{R})} + 3\|\sigma_N\|_{l_R^2}^2\right)\right] \phi_{k,xx}^{(N)2} dx\, dt\right]$$
$$\leq \left\|\partial_x \phi_0^{(N)}\right\|^2 + C_R \int_0^T (1+t)^{-2}\left(\left\|\phi_k^{(N)}\right\|_{H^1(\mathbb{R})}^2 + 1\right) dt. \tag{36}$$

Key point of the proof.

Taking derivative on (34) and implementing the L^2 estimate, the "bad" term $\|\phi_{k,x}^{(N)}\|_{L^4(\mathbb{R})}^4$ lies on the critical order in L^p interpolation inequalities for all $2 \leq p < \infty$. But using $p = \infty$ (maximal principle), we have

$$\begin{aligned}
\|\phi_{k,x}^{(N)}\|_{L^4(\mathbb{R})}^4 &\lesssim \int_{\mathbb{R}} \phi_{k,x}^{(N)2} \left|\phi_{k,xx}^{(N)}\right| dx \\
&\leq \frac{1}{\lambda} \int_{\mathbb{R}} (\phi_{k,x}^{(N)})^4 dx + \lambda \left\|\phi_{k,xx}^{(N)}\right\|^2 \\
&= -\frac{3}{\lambda} \int_{\mathbb{R}} \phi_k^{(N)} (\phi_{k,x}^{(N)})^2 \phi_{k,xx}^{(N)} dx + \lambda \left\|\phi_{k,xx}^{(N)}\right\|^2 \\
&\leq \frac{3}{\lambda} \left\|\phi_k^{(N)}\right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} (\phi_{k,x}^{(N)})^2 \left|\phi_{k,xx}^{(N)}\right| dx + \lambda \left\|\phi_{k,xx}^{(N)}\right\|^2 \\
&\leq \frac{3R}{\lambda} \int_{\mathbb{R}} (\phi_{k,x}^{(N)})^2 \left|\phi_{k,xx}^{(N)}\right| dx + \lambda \left\|\phi_{k,xx}^{(N)}\right\|^2,
\end{aligned} \tag{37}$$

choosing $\lambda = 6R$, we have $\int_{\mathbb{R}} \phi_{k,x}^{(N)2} \left|\phi_{k,xx}^{(N)}\right| dx \leq 12R \left\|\phi_{k,xx}^{(N)}\right\|^2$. □

We need a crucial lemma to extends the regularity of $\phi^{(N)}$ to any $T > 0$.

Lemma 9

Assume (H). For any fixed $T > 0$ and N, let $\phi^{(N)} \in X_1^{(N)}(T)$ be the solution of (19) for $R > \|\phi_0\|_{L^\infty(\mathbb{R})}$, we have

$$\mathbb{P}\left(\liminf_{k\to\infty} \tau_k^{(N)}(R) < T\right) = 0. \tag{38}$$

Proof.

By the a priori L^p estimates we know that there exists a subsequence, which is still denoted by $\tau_k^{(N)}(R)$, such that

$$\begin{aligned} &\phi_k^{(N)} \overset{weakly}{\longrightarrow} \phi^{(N)}, \text{ in } L^2\left([0,T], H^1(\mathbb{R})\right) \quad \text{a.s.,} \\ &\sup_{0\le t\le T} \|\phi^{(N)}(t)\|_{L_N^\infty} \leqslant \|\phi_0\|_{L^\infty(\mathbb{R})} \quad \text{a.s..} \end{aligned} \tag{39}$$

The lemma is followed by the continuously embedding from $H^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$. □

Small inital value gives H^2 regularity

To obtain the L^∞ decay we need to improve the regularity of solution ϕ to H^2

Lemma 10 (H^2 regularity)

Assume (H) and the initial perturbation $\|\phi_0\|$ small enough, i.e., set $M := \|\phi_0\|_{L^\infty(\mathbb{R})} + \delta$, if

$$\inf_{|x|\le R} b(x) \ge 12\,\|\phi_0\|_{L^\infty(\mathbb{R})}\left(\|b'\|_{L^\infty_M} + 3\|\sigma''\|^2_{l^2(L^2_M)}\right), \tag{40}$$

then we have a unique $\phi \in H^2(\mathbb{R})$ solves the equation (19) *with the following estimate*

$$\mathbb{E}\sup_{0\le s\le T}\left\|\phi(s)\right\|^2_{H^1(\mathbb{R})} + \mu\mathbb{E}\int_0^T \|\phi_x\|^2_{H^1(\mathbb{R})}\,ds \le \|\phi_0\|^2_{H^1(\mathbb{R})} + C\ln(1+T). \tag{41}$$

Main idea of the proof: It is followed by putting the derivative estimate (36) in the convergence argument.

Decay estimate: L^p decay for general H^1 initial value, $2 \leq p < \infty$

Lemma 11 (L^p decay)

Assumen (H), let $\phi \in X_1(T)$ be the unique strong solution of (11), then for any $2 \leq p < \infty$ we have

$$\mathbb{E}\|\phi\|^p_{\mathbb{L}^p(\mathbb{R})} \leq C_p(1+t)^{-\frac{p-2}{4}} \ln^{\frac{p}{2}}(2+t). \tag{42}$$

The inequality (42) is shown by the BDG inequality and the decay property of rarefaction wave, i.e, $\bar{u}_x \leq \frac{1}{t}$.

Continue

Main difficulty in the proof of lemma 11.

BDG inequality and (25) yield that

$$\mathbb{E}\|\phi\|^p \le \mathbb{E}(\sup_{0\le s\le t}\|\phi\|^2)^{\frac{p}{2}} \le C\left(\ln^{\frac{p}{2}}(2+t) + \mathbb{E}\left[\int_0^t \sum_k \left(\int_R \phi\sigma_k(\phi+\bar{u})_x dx\right)^2 ds\right]^{\frac{p}{4}}\right)$$

$$\le C\left(\ln^{\frac{p}{2}}(2+t) + 2\|\sigma'\|^2_{l^2(H^1_M)}\,\mathbb{E}\left[\int_0^t \left(\int_R |\phi|\bar{u}_x dx\right)^2 ds\right]^{\frac{p}{4}}\right),$$

and for some $\frac{1}{2} < \beta_n < 1$ determined later,

$$\mathbb{E}\left[\int_0^t \left(\int \phi\bar{u}_x dx\right)^2 dx\right]^{\frac{p_{n+1}}{4}}$$

$$\le \mathbb{E}\left[\int_0^t \left(\int |\phi|^{\beta_n}(|\phi|^{1-\beta_n}\bar{u}_x^{\frac{1}{2}})\bar{u}_x^{\frac{1}{2}} dx\right)^2\right]^{\frac{p_{n+1}}{4}}$$

$$\le C\ln(2+t) + \ln^{\frac{\beta_n p_{n+1}}{4-(1-\beta_n)p_{n+1}}}(2+t)\mathbb{E}\sup_{0\le s\le t}\|\phi\|^{\frac{2\beta_n p_{n+1}}{4-(1-\beta_n)p_{n+1}}}(s).$$

Continue

Choosing $\beta_n = 1 - \frac{2}{3p_n}$ such that $\frac{2\beta_n p_{n+1}}{4-(1-\beta_n)p_{n+1}} = p_n$, then

$$p_{n+1} \geq \frac{3}{2} p_n$$

we obtain that

$$\begin{aligned} \mathbb{E}||\phi||^{p_{n+1}} &\leq C(\ln^{\frac{p_{n+1}}{2}}(2+t) + \ln^{\frac{p_n}{2}}(2+t)\mathbb{E}||\phi||^{p_n}) \\ &\leq C\ln^{\frac{3}{2}p_n}(2+t) \leq C\ln^{p_{n+1}}(2+t). \end{aligned} \tag{43}$$

$p \in (p_n, p_{n+1})$ can be justified by the Hölder inequality. Hence (43) is true for all $p > 2$.

Proof of theorem 1.

By lemma 6 and lemma 11 we have theorem 1. □

Decay estimate: L^∞ decay for small H^1 initial value

Use the maximal principle and time weighted estimate in the derivative estimates we have

Lemma 12 (Derivative decay)

Under the assumptions in Lemma 10, let $\phi \in X_2(T)$ be the solution of (11), we have the derivative estimate

$$\mathbb{E}||\phi_x||^2 \leq C(1+t)^{-1}\ln(1+t). \tag{44}$$

By Sobolev interpolation inequalities we have

Lemma 13 (L^∞ decay)

Under the condition of Lemma 10. Let $\phi \in X_2(T)$ be the solution of (11), then for any $p > 2$,

$$\mathbb{E}||\phi||_{\mathbb{L}^\infty(\mathbb{R})} \leq C_p(2+t)^{-\frac{1}{4}}\ln^{\frac{1}{2}}(1+t). \tag{45}$$

Proof of theorem 2.

Directly obtained by lemma 10, 12 and 13. □

Special case: for stochastic Burgers euqation,
where $f(u) = \frac{1}{2}u^2, \sigma(u) = u$, we have almost surely decay rates in our previous work.

Theorem 14 (Dong, Huang, Su 2021)

For any $\epsilon > 0$, there exists a $\mathscr{F}_\infty$ measurable random variable $C_\epsilon(\omega) \in \mathbb{L}^2(\Omega)$ such that

$$\left\|u(t,\cdot) - \bar{u}(t,\cdot)\right\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_\epsilon(\omega)(2+t)^{-\frac{1}{4}+\epsilon}, \ a.s.$$

Remark 4

For nonlinear flux σ_i, the martingal estimates in the L^p frame is difficult. We can not get the almost surely decay.

Thank you !